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Chaos and information entropy production

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Abstract. We consider a general N-degrees-of-freedom nonlinear system which is chaotic and dissipative, and show that the nature of chaotic diffusion is reflected in the correlation of fluctuation of the linear stability matrix for the equation of motion of the dynamical system whose phase space variables behave as stochastic variables in the chaotic regime. Based on a Fokker–Planck description of the system in the associated tangent space and an information entropy balance equation, a relationship between chaotic diffusion and the thermodynamically inspired quantities such as entropy production and entropy flux is established. The theoretical propositions have been verified by numerical experiments.

1. Introduction

Several authors have enquired recently concerning the relationship between the phase space dynamics of a dynamical system and thermodynamics [1–9]. The question acquires a particular relevance for the dissipative system when the phase space volume contracts by virtue of possessing the attractors and also when the system is nonlinear and comprises a few-degrees-of-freedom. Thus even when these systems are not truly statistical in the thermodynamic sense, it is possible that chaotic diffusion due to intrinsic deterministic chaos or stochasticity plays a significant role in the dynamics. It is therefore worthwhile to enquire concerning the relationship between *chaotic diffusion* in a dynamical system and the *thermodynamically inspired quantities such as entropy production and entropy flux*. Our purpose in this paper is to address this specific issue.

In what follows we shall be concerned with the nonlinear dynamical systems which are chaotic and dissipative. We do not consider any stochasticity due to the thermal environment or external non-thermal noise. 'Deterministic stochasticity' (i.e. chaos) has a purely dynamical basis and its emergence in nonlinear dynamical systems is essentially due to loss of correlation of initially nearby trajectories. This is reflected in the linear stability matrix or Jacobian of the system [10]. When chaos has fully set in, the time dependence of this matrix can be described as a stochastic process, since the phase space variables behave as stochastic variables [11]. It has been shown that this fluctuation is amenable to a theoretical description in terms of the theory of multiplicative noise [12]. Based on this consideration a number of important results of non-equilibrium statistical mechanics, such as Kubo relations, the fluctuation–decoherence relation, fluctuation–dissipation relation and exponential divergence of quantum fluctuations have been realized in chaotic dynamics of a few-degrees-of-freedom system [13–18]. In the present paper we make use of this stochastic description of chaotic

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dynamics to formulate a Fokker–Planck equation of the probability density function for the relevant dynamical variables, the 'stochasticity' (i.e. chaoticity) being incorporated through the fluctuations of the time-dependent linear stability matrix. Once the drift and chaotic diffusion terms are appropriately identified the thermodynamic-like quantities can be derived with the help of the suitable information entropy balance equation.

The paper is organized as follows. In section 2 we introduce a Fokker–Planck description of the dynamical system and identify the chaotic drift and diffusion terms. This is followed by setting up of an information entropy balance equation in section 3. We then look for the entropy flux and entropy-production-like terms in the steady state. The shift of the stationary state due to additional external forcing and the associated change in entropy production is considered in section 4. We illustrate the theory in detail with the help of an example in section 5. The paper is concluded in section 6.

2. A Fokker–Planck equation for dissipative chaotic dynamics

We are concerned here with a general N-degrees-of-freedom system whose Hamiltonian is given by

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + V(\{q_i\}, t) \qquad i = 1, \ldots, N \tag{1}$$

where $\{q_i, p_i\}$ are the coordinate and momentum of the ith degrees-of-freedom, respectively, which satisfy the generic form of the equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{2}$$

We now make the Hamiltonian system dissipative by introducing $-\gamma p_i$ in the right-hand side of the second of equations (2). For simplicity we assume γ to be the same for all the N degrees of freedom. By invoking the symplectic structure of the Hamiltonian dynamics as

$$z_i = \begin{cases} q_i & \text{for} \quad i = 1, \ldots, N \\ p_{i-N} & \text{for} \quad i = N+1, \ldots, 2N \end{cases}$$

and defining I as

$$I = \begin{bmatrix} 0 & E \\ -E & -\gamma E \end{bmatrix}$$

where E is an $N \times N$ unit matrix, and 0 is an $N \times N$ null matrix, the equation of motion for the dissipative system can be written as

$$\dot{z}_i = \sum_{j=1}^{2N} I_{ij} \frac{\partial H}{\partial z_j}. \tag{3}$$

We now consider two nearby trajectories, $z_i, \dot{z}_i$ and $z_i + X_i, \dot{z}_i + \dot{X}_i$ at the same time t in a $2N$-dimensional phase space. The time evolution of separation of these trajectories is then determined by

$$\dot{X}_i = \sum_{j=1}^{2N} J_{ij}(t) X_j \tag{4}$$

in the tangent space or separation coordinate space X_i, where

$$J_{ij} = \sum_k I_{ik} \frac{\partial^2 H}{\partial z_k \partial z_j}. \tag{5}$$

Therefore, the $2N \times 2N$ linear stability matrix $\underline{J}$ assumes the following form:

$$\underline{J} = \begin{bmatrix} 0 & E \\ M(t) & -\gamma E \end{bmatrix} \tag{6}$$

where M is an $N \times N$ matrix. Note that the time dependence of the stability matrix $\underline{J}(t)$ is due to the second derivative $\frac{\partial^2 H}{\partial z_k \partial z_j}$ which is determined by the equation of motion (3). The procedure for calculation of X_i and related quantities such as Lyapunov exponents is to solve the trajectory equation (3) simultaneously with equation (4). Thus when the dissipative system described by (3) is chaotic, $\underline{J}(t)$ becomes a ('deterministically') stochastic phase space due to the fact that z_i behave as stochastic phase space variables and the equation of motion (4) in the tangent space can be interpreted as a stochastic equation [13–18].

In the next step we shall be concerned with a stochastic description of $\underline{J}(t)$ or $M(t)$. For convenience we split up M into two parts as

$$M = M_0 + M_1(t) \tag{7}$$

where M_0 is independent of variables $\{z_i\}$ and therefore behaves as a constant part and M_1 is determined by the variables $\{z_i\}$ for $i = 1, \ldots, 2N$. M_1 refers to the fluctuating part. We now rewrite the equation of motion (4) in the tangent space as

$$\dot{X} = \underline{J}X = L\left(\{X_i\}, \{z_i\}\right) \tag{8}$$

where X and L are the vectors with $2N$ components. Corresponding to (7) L in (8) can be split up again to yield

$$\dot{X} = L^0(X) + L^1(X, \{z_i(t)\})$$

or

$$\dot{X}_i = L_i{}^0(\{X_i\}) + L_i^1(\{X_i\}, \{z_i\}) \qquad i = 1, \ldots, 2N. \tag{9}$$

Equation (4) indicates that equation (8) is linear in $\{X_i\}$. Equations (4)–(6) express the fact that the first N components of L^1 are zero and the last N components of L^1 are the functions of $\{X_i\}$ for $i = 1, \ldots, N$ only. The fluctuation in L_i^1 is caused by the chaotic variables $\{z_i\}$. By defining ∇_X as differentiation with respect to components of X, i.e. $\{X_i\}$ (explicitly $X_i = \Delta q_i$ for $i = 1, \ldots, N$ and $X_i = \Delta p_i$ for $i = N+1, \ldots, 2N$) and since $L_i^1 = 0$ for $i = 1, \ldots, N$ and $L_i^1 = L_i^1(X_1, \ldots, X_N)$ for $i = N+1, \ldots, 2N$ we have $\nabla_X \cdot L^1 = \sum_{i=1}^{N}(\frac{\partial}{\partial X_i} \cdot 0) + \sum_{i=N+1}^{2N} \frac{\partial}{\partial X_i} \cdot L_i^1(X_1, \ldots, X_N) = 0$. This allows us to write the following relation (which will be used later on),

$$\nabla_X \cdot L^1 \phi(\{X_i\}) = L^1 \cdot \nabla_X \phi(\{X_i\}) \tag{10}$$

where $\phi(\{X_i\})$ is any function of $\{X_i\}$.

Note that equation (9) by virtue of (8) is a linear differential equation with multiplicative 'noise' due to $\{z_i\}$ determined by the equation of motion (3). This is the starting point of our further analysis.

Equation (9) determines a stochastic process with some given initial conditions $\{X_i(0)\}$. We now consider the motion of a representative point X in $2N$-dimensional tangent space $(X_1, \ldots, X_{2N})$ as governed by equation (9). The equation of continuity, which expresses the conservation of points determines the variation of the density function $\phi(X, t)$ in time as given by

$$\frac{\partial \phi(X, t)}{\partial t} = -\nabla_X \cdot L(t)\phi(X, t). \tag{11}$$

Expressing A_0 and A_1 as

$$A_0 = -\nabla_X \cdot L^0 \qquad \text{and} \qquad A_1 = -\nabla_X \cdot L^1 \tag{12}$$

we may rewrite the equation of continuity as

$$\frac{\partial \phi(X, t)}{\partial t} = [A_0 + \alpha A_1(t)]\phi(X, t). \tag{13}$$

It is easy to recognize that while A_0 denotes the constant part, A_1 contains the multiplicative fluctuations through the phase space variables of the dynamical system $\{z_i(t)\}$. α is a parameter introduced from outside to keep track of the order of fluctuations in the calculations. At the end we put $\alpha = 1$.

One of the main results for the linear equations of the form (13) with multiplicative noise may now be in order [12]. The average equation of $\langle \phi \rangle$ obeys $[P(X, t) \equiv \langle \phi \rangle]$,

$$\dot{P} = \left\{ A_0 + \alpha \langle A_1 \rangle + \alpha^2 \int_0^\infty \mathrm{d}\tau \, \langle\!\langle A_1(t) \exp(\tau A_0) A_1(t - \tau) \rangle\!\rangle \exp(-\tau A_0) \right\} P(X, t). \tag{14}$$

The above result is based on second-order cumulant expansion and is valid when fluctuations are small but rapid and the correlation time τ_c is short but finite, or more precisely

$$\langle\!\langle A_1(t) A_1(t') \rangle\!\rangle = 0 \qquad \text{for} \quad |t - t'| > \tau_c. \tag{15}$$

We have, in general, $\langle A_1 \rangle \neq 0$. Here $\langle\!\langle \cdots \rangle\!\rangle$ implies $\langle\!\langle \zeta_i \zeta_j \rangle\!\rangle = \langle \zeta_i \zeta_j \rangle - \langle \zeta_i \rangle \langle \zeta_j \rangle$.

Equation (14) is exact in that limit $\tau_c \to 0$. Making use of relation (12) in (11) we obtain

$$\begin{aligned} \frac{\partial P}{\partial t} = \Big\{ &-\nabla_X \cdot \boldsymbol{L}^0 - \alpha \langle \nabla_X \cdot \boldsymbol{L}^1 \rangle + \alpha^2 \int_0^\infty \mathrm{d}\tau \, \langle\!\langle \nabla_X \cdot \boldsymbol{L}^1(t) \exp(-\tau \nabla_X \cdot \boldsymbol{L}^0) \\ &\times \nabla_X \cdot \boldsymbol{L}^1(t - \tau) \rangle\!\rangle \exp(\tau \nabla_X \cdot \boldsymbol{L}^0) \Big\} P. \end{aligned} \tag{16}$$

The above equation can be transformed into the following Fokker–Planck equation ($\alpha = 1$) for the probability density function $P(X, t)$ (the details are given in the appendix):

$$\frac{\partial P(X, t)}{\partial t} = -\nabla_X \cdot FP + \sum_{i,j} \mathcal{D}_{ij} \frac{\partial^2 P}{\partial X_i \partial X_j} \tag{17}$$

where

$$F = L^0 + \langle L^1 \rangle + Q \tag{18}$$

and Q is a $2N$-dimensional vector whose components are defined by

$$Q_j = -\int_0^\infty \langle\!\langle R'_j \rangle\!\rangle \, \mathrm{d}\tau \, \det{}_1(\tau) \det{}_2(\tau). \tag{19}$$

Here the determinants $\det_1$, $\det_2$ and R'_j are given by

$$\begin{aligned} &\det{}_1(\tau) = \left| \frac{\mathrm{d}X^{-\tau}}{\mathrm{d}X} \right| \qquad \det{}_2(\tau) = \left| \frac{\mathrm{d}X}{\mathrm{d}X^{-\tau}} \right| \\ &R'_j = \sum_i L_i^1(X, t) \frac{\partial}{\partial X_i} \sum_k L_k^1(X^{-\tau}, t - \tau) \frac{\partial X_j}{\partial X_k^{-\tau}}. \end{aligned} \tag{20}$$

It is easy to recognize F as an evolution operator. Because of the dissipative perturbation we note that div $F < 0$.

The diffusion coefficient $\mathcal{D}_{ij}$ in equation (17) is defined as

$$\mathcal{D}_{ij} = \int_0^\infty \sum_k \left\langle\!\!\left\langle L_i^1(X,t) L_k^1(X^{-\tau}, t-\tau) \frac{\mathrm{d}X_j}{\mathrm{d}X_k^{-\tau}} \right\rangle\!\!\right\rangle \mathrm{d}\tau. \tag{21}$$

We have followed van Kampen's approach closely [12] to the generalized Fokker–Planck equation (17). Before concluding this section several critical remarks regarding this derivation need attention.

First, the process $M_1(t)$ determined by $\{z_i\}$ is obtained *exactly* by solving the equations of motion (3) for the chaotic motion of the system. It is therefore necessary to emphasize that we have *not assumed* any special property of noise, such as $M_1(t)$ is Gaussian or δ-correlated. We reiterate Van Kampen's emphasis in this approach.

Second, the only assumption made concerning the noise is that its correlation time τ_c is short but finite compared with the coarse-grained timescale over which the average quantities evolve. Or, in other words, the velocity changes should be small, smooth and uncorrelated after short times. This assumption, however, puts a restriction on the applicability of the present theory to a certain class of systems, for example, systems subjected to 'hard' collisions such as billiards and also molecular systems in certain non-dissipative Hamiltonian systems such as the standard map, for which the usual assumptions concerning the rapid decay of correlations and fluctuations are not valid and entropy production does not occur. Special reference may be made in this connection to the work of Zaslavsky and collaborators [19] to demonstrate that in many real systems the decay of correlation exhibits a power-law dependence, distributions admit infinite moments and the fluctuations become long lasting. Since the mathematical difficulties in dealing with the finite arbitrary correlation time of noise in a chaotic system is quite formidable, we confine ourselves in the present discussion to chaotic systems with a short but finite noise correlation time.

Third, we take care of fluctuations up to second order which implies that the deterministic noise is not too strong.

Equation (17) is the required Fokker–Planck equation in the tangent space $\{X_i\}$. However, the important point is to note that the drift and diffusion terms are determined by the phase space $\{z_i\}$ properties of the chaotic system and depend directly on the correlation function of the fluctuations of the second derivatives of the Hamiltonian (5).

3. Information entropy balance: entropy production

We shall now consider the well known relation between the probability density function $P(X,t)$ and the information entropy S as given by

$$S = -\int \mathrm{d}X\, P(X,t) \ln P(X,t). \tag{22}$$

Note that in the above definition of entropy we use $P(X,t)$, the probability distribution function in the tangent space, since one is concerned here with the expansion of the phase space in terms of a tangent space and dilation coefficients of the dynamical system for which the expanding and contracting manifolds can be defined. On the other hand, it is worthwhile to recall the dynamic entropy of a dynamical system (Kolmogorov entropy) defined in terms of the properties of evolution in the tangent space. A remark on the connection between entropy and expansion by Sinai [20] is noteworthy in this context; 'It already seems clear that positiveness of the entropy and presence of mixing is related to extreme instability of the motion of the system: trajectories emanating from the nearby points must, generally speaking, diverge with exponential velocity. Thus entropy is characterized here by the speed of approach of the asymptotic trajectories' which is formalized by defining the expansion coefficient as a logarithm of the relative increase under the flow of a volume element in the expanding manifold. Our definition of information entropy (22) makes use of the tangent space description of the systems in terms of a logarithm of the probability of expansion in the tangent space, keeping in mind that $-\ln P$ is 'a measure of unexpectedness of an event (the amount of information) and the information entropy is a mean value of this unexpectedness for the entire system' [21]. The definition (22) is therefore different from Kolmogorov entropy. We emphasize that even in the absence of any direct formal connection between $P(X, t)$ and the phase space distribution function it is possible to use the distribution function $P(X, t)$ defined in the tangent space to have an explicit expression for an entropy-production-like quantity as a function of the properties of phase space variables $\{z_i\}$ of the dynamical system (i.e. in terms of drift and diffusion coefficients of the Fokker–Planck equation).

The above definition of an information entropy-like quantity allows us to have an evolution equation for entropy. To this end we observe from equations (17) and (22) that [22, 23]

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\int \mathrm{d}X \left[-\sum_i \frac{\partial}{\partial X_i}(F_i P) + \sum_i \sum_j \int \mathcal{D}_{ij} \frac{\partial^2 P}{\partial X_i \partial X_j} \right] \ln P. \tag{23}$$

Note that the probability density function $P(X, t)$ is defined in the tangent space $\{X_i\}$. $\mathcal{D}$ and F as expressed in equations (21) and (18), respectively, are determined by the correlation functions of fluctuations of the second derivative of the Hamiltonian of the system. Equation (23) therefore suggests that the entropy-production-like term originating from equation (23) is likely to bear the signature of the chaotic dynamics. The relation is direct and general as is evident from the following equation (obtained after partial integration of equation (23) with the natural boundary condition on $P(X, t)$ that it vanishes as $|X| \to \infty$ and assuming the X dependence of $\mathcal{D}_{ij}$ to be weak (as a first approximation)):

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \int \mathrm{d}X\, P \nabla_X \cdot F + \sum_i \sum_j \mathcal{D}_{ij} \int \frac{1}{P} \frac{\partial P}{\partial X_i} \frac{\partial P}{\partial X_j}\, \mathrm{d}X. \tag{24}$$

The first term in (24) has no definite sign, while the second term is positive definite because of positive definiteness of $\mathcal{D}_{ij}$. Therefore, the second one can be identified [22] as the entropy production

$$S_{prod} = \sum_i \sum_j \mathcal{D}_{ij} \int \frac{1}{P_s} \frac{\partial P_s}{\partial X_i} \frac{\partial P_s}{\partial X_j}\, \mathrm{d}X \tag{25}$$

in the steady state. The subscript s of P_s refers to steady state. It is evident from equation (24) that

$$\begin{aligned} S_{flux} &= \int \mathrm{d}X\ P_s(X)\ \nabla_X \cdot F = \overline{\nabla_X \cdot F} \\ S_{prod} &= -S_{flux}. \end{aligned} \tag{26}$$

Note that since the chaotic system is dissipative $\overline{\nabla_X \cdot F}$ is negative (see equation (18)).

It is thus evident that the relations (25) and (26) illustrate the dynamical origin of an entropy-production-like quantity in a chaotic dissipative system. The dynamical signature is manifested through the drift term F and the chaotic diffusion terms in $\mathcal{D}_{ij}$. It must be emphasized that the notion of diffusion has nothing to do with any external reservoir. Rather it pertains to intrinsic diffusion in phase space of the chaotic system itself.

4. The chaotic system driven by an external force

We shall now examine the entropy production when the dissipative chaotic system is thrown away from the steady state due to an additional weak applied force. To this end we consider the drift F_1 due to an external force so that the total drift F now has two contributions:

$$F(X) = F_0(X) + hF_1(X). \tag{27}$$

When $h = 0$, $P = P_s$. The deviation of P from P_s in the presence of non-zero small h can be explicitly taken into account once we make use of the identity for the diffusion term [22]

$$\frac{\partial^2 P}{\partial X_i \partial X_j} = \frac{\partial}{\partial X_i}\left[P\frac{\partial \ln P_s}{\partial X_j}\right] + \frac{\partial}{\partial X_i}\left[P_s \frac{\partial}{\partial X_j}\frac{P}{P_s}\right]. \tag{28}$$

When $P = P_s$ the second term in (28) vanishes. In the presence of additional forcing equation (17) becomes

$$\frac{\partial P}{\partial t} = -\nabla_X \cdot \psi P - h\nabla_X \cdot F_1 P + \sum_i \sum_j \mathcal{D}_{ij}\frac{\partial}{\partial X_i}\left(P_s \frac{\partial}{\partial X_j}\frac{P}{P_s}\right) \tag{29}$$

where ψ is defined as

$$\psi = F_0 - \sum_j \mathcal{D}_{ij}\frac{\partial \ln P_s}{\partial X_j}. \tag{30}$$

Here we have assumed for simplicity that $\mathcal{D}_{ij}$ is not affected by the additional forcing. The leading-order influence is taken into account by the additional drift term in equation (29).

Under the steady-state condition ($P = P_s$) and $h = 0$, the second and the third terms in (29) vanish yielding

$$\nabla_X \cdot \psi P_s = 0. \tag{31}$$

It is immediately apparent that ψP_s refers to a current $\mathcal{J}$, where $\mathcal{J} = \psi P_s$. The steady-state condition therefore reduces to an equilibrium condition ($\mathcal{J} = 0$) if

$$\psi = 0. \tag{32}$$

(In section 5 we shall consider an explicit example to show that $\psi = 0$.) This suggests a formal relation between F_{0i} and $\mathcal{D}_{ij}$ as

$$F_{0i} = \sum_j \mathcal{D}_{ij}\frac{\partial \ln P_s}{\partial X_j} \tag{33}$$

where P_s may now be referred to as the *equilibrium* density function in separation coordinate space. F_0 contains a dissipation constant γ and the diffusion matrix $\mathcal{D}_{ij}$ is a function of the correlation function of fluctuations of the second derivative of the Hamiltonian.

To consider the information entropy balance equation in the presence of external forcing we first differentiate equation (22) with respect to time and use equation (29) to obtain

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}X\, P \ln P_s - \int \mathrm{d}X \ln \frac{P}{P_s}\Bigg[-\nabla_X \cdot \psi P - h\nabla_X \cdot F_1 P + \sum_i \sum_j \mathcal{D}_{ij}\frac{\partial}{\partial X_i}\left(P_s \frac{\partial}{\partial X_j}\frac{P}{P_s}\right)\Bigg]. \tag{34}$$

It is apparent that as P deviates from P_s, P/P_s differs from unity and the entire second integral within the parentheses [22] is of second order. Note that $\ln P_s$ in the first integral in equation (34) is a constant of motion and the integral denotes its average. The first term vanishes because it is of higher order as it involves P^2 and others. (Moreover, since in the discussion that follows we consider the steady state, this term does not contribute to the subsequent calculations.) To compute the contribution ΔS to the entropy balance due to the external forcing only we perform integration of the second, third and fourth terms by parts. We thus obtain,

$$\frac{\mathrm{d}\Delta S}{\mathrm{d}t} = h^2 \int \mathrm{d}X\, \delta P \nabla_X \cdot F_1 + h^2 \int \mathrm{d}X \left(\sum_i F_{1i} \frac{\partial \ln P_s}{\partial X_i} \right) \delta P + \sum_i \sum_j \mathcal{D}_{ij} \int \mathrm{d}X\, P \left(\frac{\partial}{\partial X_i} \ln \frac{P}{P_s} \right) \left(\frac{\partial}{\partial X_j} \ln \frac{P}{P_s} \right) \tag{35}$$

where we have put $h\delta P = P - P_s$. In the new steady state (in the presence of $h \neq 0$), the entropy-production and the flux-like terms balance each other as follows:

$$\Delta S_{prod} = -\Delta S_{flux} \tag{36}$$

with

$$\Delta S_{prod} = \sum_{i,j} \mathcal{D}_{ij} \int \mathrm{d}X\, P \left(\frac{\partial}{\partial X_i} \ln \frac{P}{P_S} \right) \left(\frac{\partial}{\partial X_j} \ln \frac{P}{P_S} \right) \tag{37}$$

and

$$\Delta S_{flux} = h^2 \int \mathrm{d}X\, \delta P \nabla_X \cdot F_1 + h^2 \int \mathrm{d}X \left(\sum_i F_{1i} \frac{\partial \ln P_S}{\partial X_i} \right) \delta P. \tag{38}$$

In the following section we shall work out a specific example to provide explicit expressions for the entropy production and some related quantities due to external forcing.

5. Applications

5.1. Entropy production in the steady state

To illustrate the theory developed above, we now choose a driven double-well oscillator system with the Hamiltonian

$$H = \tfrac{1}{2} p_1^2 + a q_1^4 - b q_1^2 + \epsilon q_1 \cos \Omega t \tag{39}$$

where p_1 and q_1 are the momentum and position variables of the system. a and b are the constants characterizing the potential. ϵ includes the effect of the coupling constant and the driving strength of the external field with frequency Ω. This model (39) has been used extensively in recent years for the study of chaotic dynamics [13–15, 24].

The dissipative equations of motion for the tangent space variables X_1 and X_2 corresponding to q_1 and p_1 (equation (8)) read as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \underline{J} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \qquad \left\{ \begin{array}{l} \Delta q_1 = X_1 \\ \Delta p_1 = X_2 \end{array} \right\} \tag{40}$$

where $\underline{J}$ as expressed in our earlier notation

$$z_1 = q_1 \qquad z_2 = p_1$$

is given by

$$\underline{J} = \begin{pmatrix} 0 & E \\ -E & -\gamma E \end{pmatrix} \begin{pmatrix} \dfrac{\partial^2 H}{\partial z_1 \partial z_1} & \dfrac{\partial^2 H}{\partial z_1 \partial z_2} \\ \dfrac{\partial^2 H}{\partial z_2 \partial z_1} & \dfrac{\partial^2 H}{\partial z_2 \partial z_2} \end{pmatrix}.$$

Therefore, $\underline{J}$ reduces to

$$\begin{pmatrix} 0 & 1 \\ \zeta(t) + 2b & -\gamma \end{pmatrix}$$

where $\zeta(t) = -12az_1^2$. Thus we have

$$M_0 = 2b \qquad M_1 = \zeta(t).$$

Equation (40) is thus rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = L^0 + L^1 \tag{41}$$

with

$$L^0 = \begin{pmatrix} X_2 \\ 2bX_1 - \gamma X_2 \end{pmatrix} \qquad \text{and} \qquad L^1 = \begin{pmatrix} 0 \\ \zeta(t) X_1 \end{pmatrix}$$

where L^0 and L^1 are the constant and the fluctuating parts, respectively. The fluctuations in L^1, i.e. in $\zeta(t)$, are due to stochasticity of the following chaotic dissipative dynamical equations of motion:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -az_1^3 + 2bz_1 - \epsilon \cos \Omega t - \gamma z_2. \end{aligned} \tag{42}$$

Now for the constant part and the fluctuating part we write

$$L^{01} = X_2 \qquad L^{02} = 2bX_1 - \gamma X_2$$
$$L^{11} = 0 \qquad L^{12} = \zeta(t) X_1.$$

We may then apply the result of equation (A5).

The mapping $X \to X^t$ is found by solving the 'unperturbed' equations

$$\dot{X}_1 = X_2$$
$$\dot{X}_2 = G_2 - \gamma X_2.$$

Comparison with equation (A7) shows that G_2 $(= 2bX_1)$ is free from X_2.

As a short-time approximation we consider the variation of X_1 and X_2 during τ_c:

$$\begin{aligned} X_1^{-\tau} &= -\tau X_2 + X_1 = \bar{G}_1(X_1, X_2) \\ X_2^{-\tau} &= -\tau G_2 + \mathrm{e}^{\gamma\tau} X_2 = \bar{G}_2(X_1, X_2). \end{aligned} \tag{43}$$

So the g-matrix of equation (A15) becomes

$$\underline{g} = \begin{pmatrix} 1 & \tau \mathrm{e}^{-\gamma\tau} \\ 2b\tau \mathrm{e}^{-\gamma\tau} & \mathrm{e}^{-\gamma\tau} \end{pmatrix}. \tag{44}$$

The vector R from equation (A15) can then be identified as

$$R = \begin{pmatrix} \zeta(t-\tau)(X_1 - \tau X_2)g_{12} \\ \zeta(t-\tau)(X_1 - \tau X_2)g_{22} \end{pmatrix} = \begin{pmatrix} \zeta(t-\tau)X_1\tau \mathrm{e}^{-\gamma\tau} \\ \zeta(t-\tau)(X_1 - \tau X_2)\mathrm{e}^{-\gamma\tau} \end{pmatrix} \tag{45}$$

(neglecting the terms of $\mathrm{O}(\tau^2)$).

Similarly, the vector R' is given by

$$R' = \begin{pmatrix} 0 \\ -\zeta(t-\tau)\tau \mathrm{e}^{-\gamma\tau} X_1 \zeta(t) \end{pmatrix}. \tag{46}$$

From equations (43) and (44) we have

$$\det{}_1(\tau)\det{}_2(\tau) \simeq 1. \tag{47}$$

Then the vector Q can be written as

$$Q = \begin{pmatrix} 0 \\ X_1 \displaystyle\int_0^\infty \langle\langle \zeta(t)\zeta(t-\tau)\rangle\rangle \tau \mathrm{e}^{-\gamma\tau}\,\mathrm{d}\tau \end{pmatrix}. \tag{48}$$

Now the diffusion matrix $\mathcal{D}$ can be constructed as

$$\mathcal{D} = \begin{pmatrix} 0 & 0 \\ \mathcal{D}_{21} & \mathcal{D}_{22} \end{pmatrix} \tag{49}$$

where

$$\mathcal{D}_{21} = X_1^2(0) \int_0^\infty \langle\langle \zeta(t)\zeta(t-\tau)\rangle\rangle \tau \mathrm{e}^{-\gamma\tau}\,\mathrm{d}\tau$$

and

$$\mathcal{D}_{22} = X_1^2(0) \int_0^\infty \langle\langle \zeta(t)\zeta(t-\tau)\rangle\rangle \mathrm{e}^{-\gamma\tau}\,\mathrm{d}\tau - X_1(0)X_2(0) \int_0^\infty \langle\langle \zeta(t)\zeta(t-\tau)\rangle\rangle \tau \mathrm{e}^{-\gamma\tau}\,\mathrm{d}\tau.$$

It is important to mention that the assumption of a weak X dependence of the diffusion coefficient (by freezing its time dependence) is permitted as a first approximation within the perview of the present second-order theory for which the strength of noise is not too large. We also emphasize that for an actual theoretical estimate of the entropy production in terms of the formulae (26) or (58), an explicit evaluation of the diffusion coefficients is not required (see the next section). A straightforward calculation of drift is sufficient for the purpose. This point will be clarified in greater detail in section 5.2.

Then the Fokker–Planck equation (17) for the dissipative driven double-well oscillator assumes the following form:

$$\frac{\partial P}{\partial t} = -X_2 \frac{\partial P}{\partial X_1} - \omega^2 X_1 \frac{\partial P}{\partial X_2} + \gamma \frac{\partial}{\partial X_2}(X_2 P) + \mathcal{D}_{21} \frac{\partial^2 P}{\partial X_2 \partial X_1} + \mathcal{D}_{22} \frac{\partial^2 P}{\partial X_2^2} \tag{50}$$

where

$$\omega^2 = 2b + c + c_2$$

with

$$c_2 = \int_0^\infty \langle\langle \zeta(t)\zeta(t-\tau)\rangle\rangle \tau \mathrm{e}^{-\gamma\tau}\,\mathrm{d}\tau$$
$$c = \langle \zeta \rangle. \tag{51}$$

The similarity of equation (50) to a generalized Kramers' equation cannot be overlooked. This suggests a clear interplay of chaotic diffusive motion and dissipation in the dynamics.

We now let

$$U = a_s X_1 + X_2 \tag{52}$$

where a_s is a constant to be determined.

Then under the steady-state condition, equation (50) reduces to the following form:

$$\frac{\partial}{\partial U}(\lambda_s U)P_s + \mathcal{D}_s \frac{\partial^2 P_s}{\partial U^2} = 0 \tag{53}$$

where

$$\mathcal{D}_s = \mathcal{D}_{22} + a_s \mathcal{D}_{21}$$

and

$$\lambda_s U = -a_s X_2 - \omega^2 X_1 + \gamma X_2.$$

Here λ_s is again a constant to be determined. Putting (52) in $\lambda_s U$ as given above and comparing the coefficients of X_1 and X_2 we obtain

$$\lambda_s a_s = -\omega^2 \qquad \text{and} \qquad \lambda_s = -a_s + \gamma.$$

The physically allowed solutions for a_s and λ_s are as follows:

$$a_s = \frac{\gamma - \sqrt{\gamma^2 + 4\omega^2}}{2} \qquad \text{and} \qquad \lambda_s = \frac{\gamma + \sqrt{\gamma^2 + 4\omega^2}}{2}. \tag{54}$$

The stationary solution of (53) P_s is given by

$$P_s = N \mathrm{e}^{-\lambda_s U^2/2\mathcal{D}_s}. \tag{55}$$

Here N is the normalization constant. By virtue of (55) ψ corresponding to equation (30) is therefore

$$\psi = \lambda_s U - \mathcal{D}_s \frac{\partial \ln P_s}{\partial U} = 0. \tag{56}$$

Since ψP_s defines a current, P_s defines a zero-current situation or an equilibrium condition.

The equilibrium solution P_s from (55) can now be used to calculate the steady-state entropy production as given by equation (25). We thus have

$$S_{prod} = \mathcal{D}_s \int_{-\infty}^{\infty} \frac{1}{P_s} \left(\frac{\partial P_s}{\partial U}\right)^2 \mathrm{d}U. \tag{57}$$

Explicit evaluation shows

$$S_{prod} = \lambda_s \tag{58}$$

where λ_s is given by equation (54).

The above result demonstrates a rather straightforward connection between the entropy-production-like quantity of a chaotic system and the dynamics through the dissipation constant γ, parameters of the Hamiltonian and correlation of fluctuations of the second derivatives of the Hamiltonian in the steady state. It is important to note that since one is concerned here with a few-degrees-of-freedom system with no explicit reservoir, temperature does not appear in the expression for the entropy-production-like term (58). The entropy production in a truly thermodynamic system and in the present case are therefore distinct.

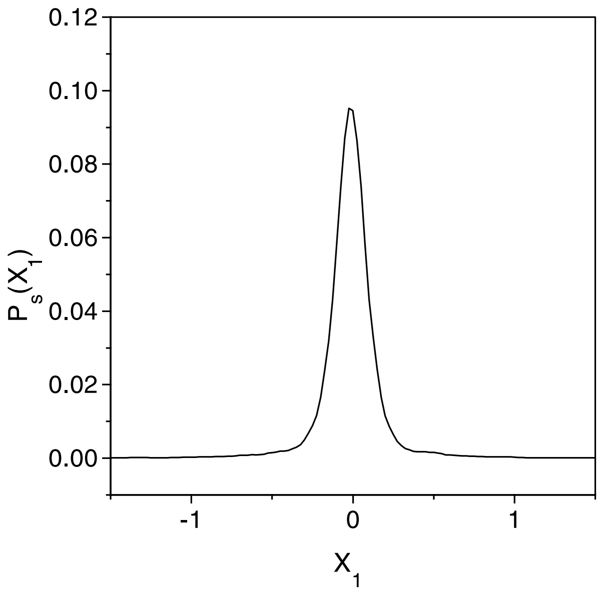

Figure 1. A plot of the numerically calculated stationary distribution function $P_s(X_1)$ as a function of X_1 for the set of parameter values described in section 5.

5.2. Numerical verifications

To verify the above theoretical analysis in terms of numerical experiments we now concentrate on the following two points. First, it is necessary to establish numerically that the dynamical system reaches a steady state, i.e. the probability density function $P(\{X_i\}, t)$ attains a steady-state distribution $P_s(\{X_i\})$ in the long-time limit. Second, the entropy production in the steady state calculated by formula (58) needs to be verified numerically. To address the first issue we now proceed as follows.

The dissipative chaotic dynamics corresponding to the model Hamiltonian (39) is governed by equations (40) and (42). We choose the following values of the parameters [24]: $a = 0.5$, $b = 10$, $\omega = 6.07$ and $\gamma = 0.001$. The coupling-cum-field strength ϵ is varied from set to set. We fix the initial condition $z_1(0) = -3.5$, $z_2(0) = 0$ which ensures strong global chaos [24]. To determine the steady-state distribution function, say, $P_s(X_1)$, where $X_1 = \Delta q$ (equation (40)) from the dynamical point of view we first define d_0 as the separation of the two initially nearby trajectories and $d(t)$ as the corresponding separation at time t. To express $d(t)$ we write $d(t) = \left[\sum_i^N (X_i)^2 + \sum_{i=N+1}^{2N} (X_i)^2\right]^{1/2}$. $d(t)$ is determined by solving equations (40) and (42) numerically, simultaneously for the initial conditions of z_1 and z_2 corresponding to equation (42). To follow the evolution of X_1 numerically in time, i.e. in going from the jth to the $(j+1)$th iteration step, say, X_1 has to be initialized as $X_1^{j0} = \frac{X_1^j}{d_j} d_0$. (The time evolution of the other components of X can be followed similarly.) This initialization implies that at each step, iteration starts with the same magnitude of d_0 but the direction of d_0 for step $j+1$ is that of $d(t)$ for the jth step (considered in terms of the ratio X_1^j/d_j). For a more pictorial illustration we refer to figure 1 of [25]. The jth iteration term means $t = jT$ ($j = 1, 2, \ldots, \infty$), where T is the characteristic time which corresponds to the shortest ensemble averaged period of the nonlinear dynamical system.

Having obtained the time series in X_1 (it may be noted that the series in X_i are also required for calculation of the largest Lyapunov exponent as defined by

$$\lambda = \lim_{\substack{n\to\infty \\ d_0\to 0}} \frac{1}{nT} \sum_{j=1}^{n} \ln \frac{d_j}{d_0}$$

for the chaotic system) the stationary probability density function $P_s(X_1)$ is computed as follows: the X_1-axis ranging from -2 to $+2$ is divided into small intervals ΔX_1 of size 0.025. The time series in X_1 is computed over the time intervals of 1000–10 000 times the time period T. For each time interval ΔX_1 a counter is maintained and is initially set to zero before the simulation is started. The respective counter is incremented whenever X_1 falls within the given interval. Finally, the steady-state probability distribution function $P_s(X_1)$ is obtained by normalizing the counts. The result is shown in figure 1 for $\epsilon = 10$. Our numerical analysis shows that the distribution function attains stationarity at around $t = 1000T$, beyond which no perceptible change in the distribution is obtained.

We now turn to the second issue. In what follows we shall be concerned with steady-state entropy production (58) and its numerical verification. This quantity can be calculated in two different ways. First, it may be noted that the determination of S_{prod} (equation (57)) rests on two quantities defined in the tangent space; the steady-state probability distribution function $P_s(U)$ and the diffusion coefficient $\mathcal{D}_s$ in U-space. Once the procedure for calculation of the distribution function as illustrated above is known from the time series in X_1 or X_2 the evaluation of $P_s(U)$ is quite straightforward since U is expressed as $U = a_s X_1 + X_2$ according to (52). Here a_s is given by (54) with $\omega^2 = 2b + c + c_2$ and the average c and the integral over the correlation function, c_2 are as defined in (51). To calculate the correlation function $\langle\langle\zeta(t)\zeta(t-\tau)\rangle\rangle$ and the average $\langle\zeta(t)\rangle$ it is necessary to determine long time series in $\zeta(t)$ ($\zeta(t) = -12az_1^2$) by solving numerically the classical equation of motion (42) in phase space followed by averaging over the time series. For further details of numerical analysis of correlation functions we refer to the earlier work [15–17]. The diffusion coefficient $\mathcal{D}_s$ can be determined numerically from the time series in U. Knowing $P_s(U)$ and $\mathcal{D}_s$, one can make use of formula (57) to obtain the entropy production in the steady state. S_{prod} is thus calculated numerically. The second procedure of calculation of S_{prod} is the direct theoretical evaluation of λ_s from the expression (54). Since the value of λ_s again rests on ω^2 and γ, and the dependence of ω^2 on the averages and the correlation functions are already known from the numerical analysis of phase space, λ_s can be calculated in the usual way. In figure 2 we compare the values of steady-state entropy production thus obtained by the two different methods for several values of coupling-cum-field strength ϵ. Here it should be noted that the curve connecting the squares (i.e. the theoretically calculated entropy production in the steady state) corresponds to the negative of the entropy flux (S_{flux}) since for the given example S_{flux} in U space is $-\lambda_s$ ($-\int \lambda_s P_s(U)\,\mathrm{d}U$, from equation (26)) for the normalized probability distribution function $P_S(U)$. Thus figure 2 is a numerical proof of $S_{prod} = -S_{flux}$. The agreement is found to be quite satisfactory. We therefore conclude that at least for the model studied here and for the similar class of models the correspondence between the formulae of steady-state entropy production and the numerical computation is fairly general.

5.3. *Entropy production in presence of weak forcing*

We now introduce an additional weak forcing in the dynamics. This is achieved by subjecting the dissipative chaotic system to a weak magnetic field ($\vec{B}$) through a velocity-($\vec{v}$) dependent force term $\frac{e}{c_l}\vec{v}\times\vec{B}$, where e and c_l are the electric charge and the velocity of light, respectively. For simplicity we apply the constant field B_z which is perpendicular to the q_1-direction (see

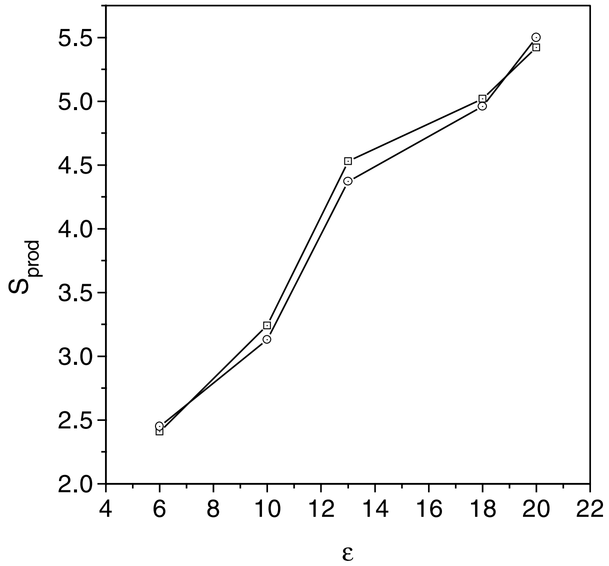

Figure 2. The steady entropy production calculated numerically (circle) and theoretically using equation (58) (square) for different values of the driven field strength ϵ for the model described in section 5.

equation (39)). In the presence of this force field the motion of the particle will not be restricted to B_z and q_1 only. We have to consider the other direction q_2 which is perpendicular to both q_1 and B_z.

To make the notation consistent with equation (8) we would now like to let X_1, X_2, X_3 and X_4 correspond to Δq_1, Δq_2, Δp_1 and Δp_2, respectively.

The relevant equations of motion are therefore as follows:

$$\begin{aligned}
\dot{X}_1 &= X_3 \\
\dot{X}_2 &= X_4 \\
\dot{X}_3 &= 2bX_1 - \gamma X_3 + \zeta(t)X_1 + h\frac{eX_4}{c_l}B_z \\
\dot{X}_4 &= -h\gamma X_4 - h\frac{eX_3}{c_l}B_z.
\end{aligned} \tag{59}$$

Here e/c_l is the ratio of electric charge to the velocity of light used to give the equation the appropriate dimensions.

Then the non-equilibrium situation (due to additional forcing, $h \neq 0$) corresponding to equation (59) is governed by

$$\begin{aligned}
\frac{\partial P}{\partial t} = &-h\frac{\partial}{\partial X_2}(X_4P) + h\frac{\partial}{\partial X_4}(\gamma X_4)P - hB'_z\frac{\partial}{\partial X_3}(X_4P) \\
&+hB'_z\frac{\partial}{\partial X_4}(X_4P) + \mathcal{D}_s\frac{\partial}{\partial U}\left(P_s\frac{\partial}{\partial U}\frac{P}{P_s}\right)
\end{aligned} \tag{60}$$

where

$$\begin{aligned}
B'_z &= \frac{e}{c_l}B_z \\
\mathcal{D}_s &= \mathcal{D}_{22} + \mathcal{D}_{21}a_s
\end{aligned}$$

or more explicitly,

$$\frac{\partial P}{\partial t} = -X_3\frac{\partial P}{\partial X_1} - \omega^2 X_1\frac{\partial P}{\partial X_3} + \gamma\frac{\partial}{\partial X_3}(X_3P) + h\gamma\frac{\partial}{\partial X_4}(X_4P) - hX_4\frac{\partial P}{\partial X_2}$$
$$-hB_z'\frac{\partial}{\partial X_3}(X_4P) + hB_z'\frac{\partial}{\partial X_4}(X_3P) + \mathcal{D}_{21}\frac{\partial^2 P}{\partial X_3\partial X_1} + \mathcal{D}_{22}\frac{\partial^2 P}{\partial X_3^2}. \tag{61}$$

Proceeding as before we make use of the transformation of variables using

$$U' = a'X_1 + b'X_2 + c'X_3 + X_4 \tag{62}$$

so that equation (61) becomes

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial U'}(\lambda' U' P) + \mathcal{D}\frac{\partial^2 P}{\partial U'^2} \tag{63}$$

where

$$\mathcal{D} = \mathcal{D}_{33}c'^2 + \mathcal{D}_{31}a'c'. \tag{64}$$

$\mathcal{D}_{33}$ and $\mathcal{D}_{31}$ in four dimensions correspond to $\mathcal{D}_{22}$ and $\mathcal{D}_{21}$ in two dimensions, respectively and

$$\lambda' U' = -a'X_3 - c'\omega^2 X_1 + \gamma X_3 c' + h\gamma X_4 - hB_z' X_4 c' + hB_z' X_3 - hb' X_4. \tag{65}$$

Using (62) in (65) and comparing the coefficients of X_i we obtain

$$b' = 0 \qquad a' = c'\gamma + hB_z' - \lambda' c'$$
$$c' = (h\gamma - \lambda')/hB_z' \tag{66}$$

where λ' is a solution of the cubic algebraic equation

$$\lambda'^3 + \lambda'(h\gamma^2 + h^2 B_z'^2 - \omega^2) - \lambda'^2\gamma(1+h) + \omega^2\gamma h = 0. \tag{67}$$

We now seek a perturbative solution of the algebraic equation (67) which is given by (h as a small parameter)

$$\lambda' = \lambda_0' + \frac{h\left(\lambda_0'^2\gamma - \omega^2\gamma - \lambda_0'\gamma^2 - \lambda'_0 hB_z'^2\right)}{3\lambda_0'^2 - 4\lambda_0'\gamma + h\gamma^2 + h^2B_z'^2 - \omega^2} \tag{68}$$

where λ_0' is the solution of (67) for $h = 0$;

$$\lambda_0' = \frac{\gamma + \sqrt{\gamma^2 + 4\omega^2}}{2}. \tag{69}$$

This is identical to λ_s (equation (54)). Therefore, by virtue of equations (66)–(69) all the constants in (62), i.e. a', b', c' are now known. The stationary solution of (63) is now given by

$$P_s' = N'\mathrm{e}^{-\lambda' U'^2/2\mathcal{D}} \tag{70}$$

where N' is the normalization constant.

We are now in a position to calculate the steady-state entropy flux ΔS_{flux} due to external forcing ($h \neq 0$) from equation (38)

$$\Delta S_{flux} = h^2\int \mathrm{d}X\,\delta P\nabla\cdot F_1 + h^2\int \mathrm{d}X\left(\sum_i F_{1i}\frac{\mathrm{d}\ln P_s}{\mathrm{d}X_i}\right)\delta P \tag{71}$$

where the components of F_1 can be identified as

$$\begin{aligned} &F_{11} = 0 && F_{13} = B_z' X_4 \\ &F_{12} = X_4 && F_{14} = -B_z' X_3 - \gamma X_4 \\ &\nabla_X \cdot F_1 = -\gamma, \end{aligned} \tag{72}$$

and $h\delta P = P_s' - P_s$ denotes the deviation from the initial equilibrium state due to external forcing. For normalized probability functions P_s' and P_s the first integral in (71) vanishes.

Since P_s is given by (55) with U as defined in (52), the expression for (71) reduces to

$$\begin{aligned} \Delta S_{prod} &= -\Delta S_{flux} \\ &= h B_z' \frac{\lambda_s}{\mathcal{D}_s} \int X_4 \, (a_s X_1 + X_3) \, \mathrm{d}X \, P_s'. \end{aligned} \tag{73}$$

We now use the following transformations of variables:

$$\begin{aligned} &u' = a' X_1 + c' X_3 + X_4 \qquad (\text{since } b' = 0) \\ &v' = X_3 \\ &w' = X_4 \\ &\mathrm{d}X_1 \, \mathrm{d}X_3 \, \mathrm{d}X_4 = a' \, \mathrm{d}u' \, \mathrm{d}v' \, \mathrm{d}w' \end{aligned} \tag{74}$$

to calculate the integrals,

$$\int X_3 X_4 \, \mathrm{d}X \, P_s' = \frac{\mathcal{D}}{2\lambda' |c'|} \qquad \text{and} \qquad \int X_1 X_4 \, \mathrm{d}X \, P_s' = \frac{\mathcal{D}}{2\lambda' a'} \tag{75}$$

which yield

$$\begin{aligned} \Delta S_{prod} &= -\Delta S_{flux} \\ &= h B_z' \frac{\lambda_s}{\mathcal{D}_s} \frac{\mathcal{D}}{2\lambda'} \left[\frac{1}{|c'|} + \frac{|a_s|}{a'} \right]. \end{aligned} \tag{76}$$

For numerical verification of the above theoretical result (76) one can calculate entropy production (ΔS_{prod}) numerically in the steady state in the presence of weak forcing from equation (37) as in the previous subsection. Equation (37) for the present example reduces to the following form in the steady state:

$$\begin{aligned} \Delta S_{prod} = \mathcal{D}_{33} &\iiint P_s' \left(\frac{\partial}{\partial X_3} \ln \frac{P_s'}{P_s} \right)^2 \mathrm{d}X_1 \, \mathrm{d}X_3 \, \mathrm{d}X_4 \\ &+ \mathcal{D}_{31} \iiint P_s' \left(\frac{\partial}{\partial X_3} \ln \frac{P_s'}{P_s} \right) \left(\frac{\partial}{\partial X_1} \ln \frac{P_s'}{P_s} \right) \mathrm{d}X_1 \, \mathrm{d}X_3 \, \mathrm{d}X_4. \end{aligned} \tag{77}$$

To calculate ΔS_{prod} numerically, $\mathcal{D}_{33}$, $\mathcal{D}_{31}$ and P_s can be determined by directly using the procedure mentioned in subsection 5.2 and by solving equations (41) and (42) simultaneously. Similarly, one can calculate P_s' from equations (42) and (59). Finally, making use of all of these quantities in equation (77) ΔS_{prod} can be obtained. Thus the numerically evaluated ΔS_{prod} should correspond to the results of equation (76) since our numerical verification in figure 2 shows good agreement between numerical and theoretical results, $\mathcal{D}_{33}$, $\mathcal{D}_{31}$ being very close to $\mathcal{D}_{22}$ and $\mathcal{D}_{21}$, respectively, since h is very small.

In the limit where h and γ are small the above expression (76) can be simplified further. To this end we first note that

$$\begin{aligned} &|a_s| \sim \omega \qquad \lambda_s \sim \omega \qquad \lambda' \sim \lambda_0' \sim \omega \\ &c' = \frac{a'}{\omega} \qquad \text{and} \qquad a' = \frac{\omega h}{\gamma} B_z'. \end{aligned} \tag{78}$$

This reduces $\mathcal{D}$ further as follows;

$$\begin{aligned}\mathcal{D} &= \mathcal{D}_{33}c'^2 + \mathcal{D}_{31}a'c' \\ &\simeq \frac{a'^2}{\omega^2}(\mathcal{D}_{33} - \mathcal{D}_{31}\omega). \end{aligned} \tag{79}$$

Thus we have

$$\frac{\mathcal{D}}{\mathcal{D}_s} = \frac{a'^2}{\omega^2}. \tag{80}$$

Making use of (78)–(80), expression (76) can be approximated as

$$\begin{aligned}\Delta S_{prod} &= hB'_z\frac{a'^2}{\omega^2}\frac{1}{2}\left(\frac{\omega}{a'} + \frac{\omega}{a'}\right) \\ &= \frac{h^2e^2}{c_l^2\gamma}B_z^2. \end{aligned} \tag{81}$$

This expression is due to the average of the work per unit time of the external force B_z acting on the chaotic system. Note that the quadratic dependence on the magnetic field B_z in equation (81) is characteristic of an expression for entropy production in the steady state. Since the system is not thermostated this is independent of the temperature. Although the leading-order expression (81) is apparently free from diffusion coefficients, a close look into the more exact expression (76) reveals that their influence is quite significant in the higher order.

6. Conclusions

Ever since the development of the theory of chaos, the dynamical variables in the strong chaotic regime have been interpreted as stochastic variables. One of the earliest well known examples in this connection was set by demonstrating [11] the linear divergence of mean-square momentum in time in a standard map, mimicking the Brownian motion. In the present paper we have tried to relate this chaotic diffusion to thermodynamic-like quantities by establishing a generalized Fokker–Planck equation pertaining to the tangent space. The explicit dependence of drift and diffusion terms on the dynamical characteristics of the phase space of the system is demonstrated.

The main conclusions of our study are the following.

(a) We analyse the nature of chaotic diffusion in terms of the properties of the phase space of chaotic systems. The drift and diffusion terms are dependent on the correlation of fluctuations of the linear stability matrix of the equation of motion. Since the latter is the key point for understanding the stability of motion in a dynamical system, we emphasize that the thermodynamic-like quantities as discussed here have a deeper root in the intrinsic nature of motion of a few-degrees-of-freedom system.

(b) We identify the information entropy flux and production-like terms in the steady state which explicitly reveal their connection to dynamics through drift and diffusion terms, in the presence and absence of the external force field.

(c) The connection between the thermodynamically inspired quantities and chaos are fairly general for the N-degrees-of-freedom systems.

The theory developed in this paper is based on the derivation of the Fokker–Planck equation for chaotic systems pertaining to the processes with correlation time which is short but finite (i.e. for the systems with hard chaos). The suitable generalization of the approach to more general cases, where one encounters long correlation times is worth further investigation in this direction.

Acknowledgment

BCB is indebted to the Council of Scientific and Industrial Research for a fellowship.

Appendix. The derivation of the Fokker–Planck equation

We first note that the operator $\exp(-\tau\nabla_X \cdot \boldsymbol{L}^0)$ provides the solution of the equation (equation (13), $\alpha = 0$)

$$\frac{\partial f(X,t)}{\partial t} = -\nabla_X \cdot \boldsymbol{L}^0 f(X,t) \tag{A1}$$

where f signifies the 'unperturbed' part of P, which can be found explicitly in terms of characteristic curves. The equation

$$\dot{X} = \boldsymbol{L}^0(X) \tag{A2}$$

determines for a fixed t a mapping from $X(\tau = 0)$ to $X(\tau)$, i.e. $X \to X^\tau$ with inverse $(X^\tau)^{-\tau} = X$. The solution of (A1) is

$$f(X,t) = f(X^{-t},0)\left|\frac{\mathrm{d}X^{-t}}{\mathrm{d}X}\right| = \exp\left[-t\nabla_X \cdot \boldsymbol{F}_0\right] f(X,0) \tag{A3}$$

and $\left|\frac{\mathrm{d}(X^{-t})}{\mathrm{d}(X)}\right|$ is a Jacobian determinant. The effect of $\exp(-t\nabla_X \cdot \boldsymbol{L}^0)$ on $f(X)$ is

$$\exp(-t\nabla_X \cdot \boldsymbol{L}^0) f(X,0) = f(X^{-t},0)\left|\frac{\mathrm{d}X^{-t}}{\mathrm{d}X}\right|. \tag{A4}$$

In equation (16) this simplification yields

$$\frac{\partial P}{\partial t} = \Bigg\{-\nabla_X \cdot \boldsymbol{L}^0 - \alpha\langle\nabla_X \cdot \boldsymbol{L}^1\rangle + \alpha^2\int_0^\infty \mathrm{d}\tau \left|\frac{\mathrm{d}X^{-\tau}}{\mathrm{d}X}\right| \times\langle\langle\nabla_X \cdot \boldsymbol{L}^1(X,t)\nabla_{X^{-\tau}} \cdot \boldsymbol{L}^1(\boldsymbol{x}^{-\tau}, t-\tau)\rangle\rangle \left|\frac{\mathrm{d}X}{\mathrm{d}X^{-\tau}}\right|\Bigg\} P. \tag{A5}$$

Now to express the Jacobian, $X^{-\tau}$ and $\nabla_{X^{-\tau}}$ in terms of ∇_X and X we solve equation (A2) for a short time (this is consistent with the assumption that the fluctuations are rapid [12]). Using equations (4)–(6) we may rewrite the 'unperturbed' equation (A2) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix} = \begin{pmatrix} X_{N+1} \\ \vdots \\ X_{2N} \end{pmatrix} \tag{A6}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} X_{N+1} \\ \vdots \\ X_{2N} \end{pmatrix} = -\gamma\begin{pmatrix} X_{N+1} \\ \vdots \\ X_{2N} \end{pmatrix} + \begin{pmatrix} G_{N+1}(X) \\ \vdots \\ G_{2N}(X) \end{pmatrix}. \tag{A7}$$

Here $G_{N+1}(X)\cdots G_{2N}(X)$ are functions of $\{X_i\}$ with $i = 1,\ldots,N$ only. This allows us to rewrite the solution of (A6) and (A7) as

$$\begin{pmatrix} X_1^{-\tau} \\ \vdots \\ X_N^{-\tau} \end{pmatrix} = -\tau\begin{pmatrix} X_{N+1} \\ \vdots \\ X_{2N} \end{pmatrix} + \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix} = \begin{pmatrix} \bar{G}_1(X) \\ \vdots \\ \bar{G}_N(X) \end{pmatrix} \tag{A8}$$

and

$$\begin{pmatrix} X_{N+1}^{-\tau} \\ \vdots \\ X_{2N}^{-\tau} \end{pmatrix} = \mathrm{e}^{\gamma\tau} \begin{pmatrix} X_{N+1} \\ \vdots \\ X_{2N} \end{pmatrix} - \tau \begin{pmatrix} G_{N+1}(X) \\ \vdots \\ G_{2N}(X) \end{pmatrix} = \begin{pmatrix} \bar{G}_{N+1}(X) \\ \vdots \\ \bar{G}_{2N}(X) \end{pmatrix}. \tag{A9}$$

Here the terms of $\mathrm{O}(\tau^2)$ are neglected. Since the vector $X^{-\tau}$ is expressible as a function of X we write

$$X^{-\tau} = \bar{G}(X) \tag{A10}$$

and the following simplification holds good:

$$\begin{aligned} L^1(X^{-\tau}, t-\tau) \cdot \nabla_{X^{-\tau}} &= L^1(\bar{G}(X), t-\tau) \cdot \nabla_{X^{-\tau}} \\ &= \sum_k L_k^1(\bar{G}(X), t-\tau) \frac{\partial}{\partial X_k^{-\tau}} \\ &= \sum_j \sum_k L_k^1(\bar{G}(X), t-\tau) g_{jk} \frac{\partial}{\partial X_j} \qquad j, k = 1, \ldots, 2N \end{aligned} \tag{A11}$$

where

$$g_{jk} = \frac{\partial X_j}{\partial X_k^{-\tau}}. \tag{A12}$$

In view of equations (A8) and (A9) we note:

$$\begin{aligned} &\text{if} \quad j = k \quad \text{then} \quad g_{jk} = 1 && k = 1, \ldots, N \\ &\qquad\qquad\qquad\qquad\quad = \mathrm{e}^{-\gamma\tau} && k = N+1, \ldots, 2N \\ &\text{if} \quad j \neq k \quad \text{then} \quad g_{jk} \propto -\tau \mathrm{e}^{-\gamma\tau} \\ &\qquad\qquad\qquad\qquad\quad \text{or } 0. \end{aligned}$$

Thus g_{jk} is a function of τ only.

Let

$$R_j = \sum_k L_k^1(\bar{G}(X), t-\tau) g_{jk}. \tag{A13}$$

From equations (8), (9) and (A10) we write

$$L_i^1(X^{-\tau}, t-\tau) = L_i^1(\bar{G}(X), t-\tau) = 0 \qquad \text{for} \quad i = 1, \ldots, N. \tag{A14}$$

So the conditions (A13), (A14) and (A8) imply that

$$\begin{aligned} R_j(X, t-\tau) &= R_j(X_1, \ldots, X_N, t-\tau) && \text{for} \quad j = 1, \ldots, N \\ R_j(X, t-\tau) &= R_j(X_1, \ldots, X_{2N}, t-\tau) && \text{for} \quad j = N+1, \ldots, 2N. \end{aligned} \tag{A15}$$

We next carry out the following simplifications of the α^2-term in equation (A5). We make use of relation (10) to obtain

$$\begin{aligned} L^1(X, t) \cdot \nabla \sum_j R_j \frac{\partial}{\partial X_j} P(X, t) &= \sum_i L_i^1(X, t) \frac{\partial}{\partial X_i} \sum_j R_j \frac{\partial}{\partial X_j} P(X, t) \\ &= \sum_{i,j} L_i^1(X, t) R_j \frac{\partial^2}{\partial X_i \partial X_j} P(X, t) + \sum_j R_j' \frac{\partial}{\partial X_j} P(X, t) \end{aligned} \tag{A16}$$

where

$$R'_j = \sum_i L_i^1(X,t)\frac{\partial}{\partial X_i} R_j. \tag{A17}$$

Conditions (A14) and (A15) imply that

$$\begin{aligned} R'_j &= 0 && \text{for} \quad j = 1, \ldots, N \\ R'_j &= R'_j(X_1, \ldots, X_N, t-\tau) \neq 0 && \text{for} \quad j = N+1, \ldots, 2N. \end{aligned} \tag{A18}$$

By (A18) one has

$$R' \cdot \nabla_X P(X,t) = \nabla_X \cdot R' P(X,t). \tag{A19}$$

Making use of equations (10), (A11), (A16) and (A19) in equation (A5) we obtain the Fokker–Planck equation (17).

References

[1] Kai T and Tomita K 1980 *Prog. Theor. Phys.* **64** 1532
[2] Takahashi Y and Oono Y 1984 *Prog. Theor. Phys.* **71** 851
[3] Grassberger P and Procaccia I 1984 *Physica* D **13** 34
[4] Bonci L, Roncaglia R, West B J and Grigolini P 1993 *Phys. Rev. Lett.* **67** 2593
[5] Berdichevshy V L and Alberti M V 1991 *Phys. Rev.* A **44** 858
[6] Pattanayak A K and Brumer P 1997 *Phys. Rev. Lett.* **79** 4131
[7] Ruelle D 1996 *J. Stat. Phys.* **85** 1
[8] Cohen D 1997 *Phys. Rev. Lett.* **78** 2878
[9] Gaspard P 1997 *J. Stat. Phys.* **88** 1215
[10] Fox R F and Elston T C 1994 *Phys. Rev.* E **49** 3683
[11] Casati G, Chirikov B V, Izrailev F M and Ford J 1979 *Stochastic Behaviour in Classical and Hamiltonian Systems (Lecture Notes in Physics 83)* ed G Casati and J Ford (Berlin: Springer)
[12] van Kampen N G 1976 *Phys. Rep.* **24** 171
[13] Chaudhuri S, Gangopadhyay G and Ray D S 1993 *Phys. Rev.* E **47** 311
[14] Chaudhuri S, Gangopadhyay G and Ray D S 1995 *Phys. Rev.* E **52** 2262
[15] Chaudhuri S, Gangopadhyay G and Ray D S 1996 *Phys. Rev.* E **54** 2359
[16] Bag B C, Chaudhuri S, Ray Chaudhuri J and Ray D S 1999 *Physica* D **125** 47
[17] Bag B C and Ray D S 1999 *J. Stat. Phys.* **96** 271
[18] Bag B C and Ray D S 2000 *Phys. Rev.* E **62** 1927
[19] Zaslavsky G M 1999 *Phys. Today* **52** 39
[20] Sinai Ja G 1963 *Am. Math. Transl.* **31** 62 (see p 72)
[21] See, for example, Vasilyev A M 1983 *An Introduction to Statistical Physics* (Moscow: Mir) p 53
[22] Daems D and Nicolis G 1999 *Phys. Rev.* E **59** 4000
[23] Compte A and Jou D 1996 *J. Phys. A: Math. Gen.* **29** 4321
[24] Lin W A and Ballentine L E 1990 *Phys. Rev. Lett.* **65** 2927
[25] Benettin G, Galgani L and Strelcyn J 1976 *Phys. Rev.* A **14** 2338